第三章 非线性方程求解

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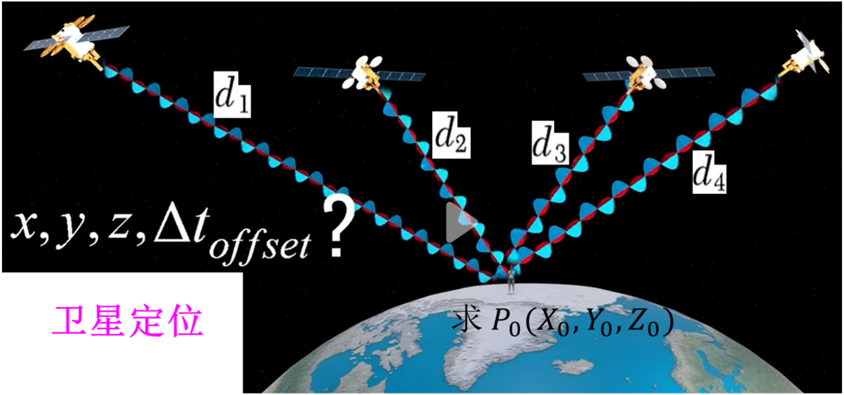
非线性方程求解

非线性科学是当今科学发展的一个很重要的研究方向，而非线性方程求根(求解)也成了其不可或缺的重要内容. 与线性方程相比，非线性方程求解问题无论是从理论上还是计算上都要复杂得多.



卫星定位

已知 $P_1(X_1, Y_1, Z_1)$ $P_2(X_2, Y_2, Z_2)$ P_3 $P_4(X_4, Y_4, Z_4)$

求 $P_0(x_0, y_0, z_0)$

$$\begin{cases}(x_0-X_1)^2+(y_0-Y_1)^2+(z_0-Z_1)^2=d_1^2(\Delta t_1,\varepsilon)\\(x_0-X_2)^2+(y_0-Y_2)^2+(z_0-Z_2)^2=d_2^2(\Delta t_2,\varepsilon)\\(x_0-X_3)^2+(y_0-Y_3)^2+(z_0-Z_3)^2=d_3^2(\Delta t_3,\varepsilon)\\(x_0-X_4)^2+(y_0-Y_4)^2+(z_0-Z_4)^2=d_4^2(\Delta t_4,\varepsilon)\end{cases}$$

$P_1(X_1,Y_1,Z_1)$ $P_2(X_2,Y_2,Z_2)$ P_3 $P_4(X_4,Y_4,Z_4)$

X= [1.4832308660e+007, -1.5799854050e+007, 1.984818910e+006, -1.2480273190e+007]
Y = [2.0466715890e+007, -1.3301129170e+007, -1.1867672960e+007, -2.3382560530e+007]
Z = [7.428634750e+006, 1.7133838240e+007, 2.3716920130e+007, 3.278472680e+006]

d = [2.4310764064e+007, 2.2914600784e+007, 2.0628809405e+007, 2.3422377972e+007]

d_1 d_2 d_3 d_4

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$$e^x - 1 = cos(\pi x) \quad \Rightarrow \text{精确解？}$$

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- 通常用迭代法求解非线性方程（组）

实根的对分法

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原理：利用连续函数的介值定理

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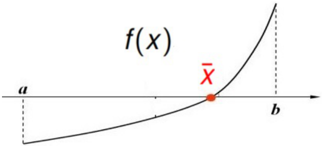

原理：利用连续函数的介值定理

$$\left[f(a)\cdot f(b)<0 \Longrightarrow \exists \bar{x}\in(a,b), s.t., f(\bar{x})=0 \right]$$

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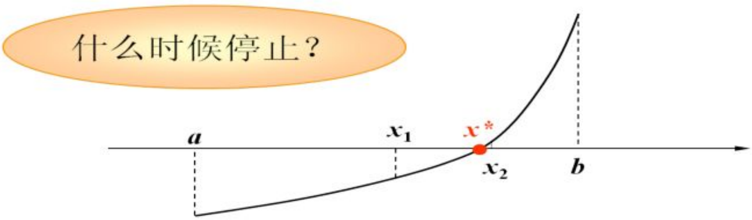

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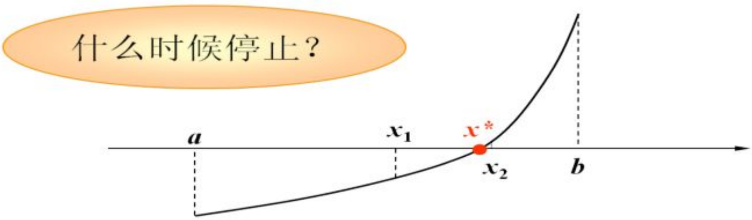

$$|x_{k+1} - x_k| < \epsilon_1 \quad \text{or} \quad |f(x_{k+1})| < \epsilon_2$$

对分法的算法描述

输入：单变元函数$f(x)$和区间$[a,b]$（满足$f(a)f(b)<0$)，精度ε.

输出：f在[a,b]上的一个近似根x^*(若存在).

While $|a-b|>\varepsilon$

$x^* := (a+b)/2$；

计算$f(x^*)$；

若 $|f(x^*)|<\varepsilon$，x^*为解，结束；

若$f(x^*)\cdot f(b)<0$，$[a,b] := [x^*,b]$；

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End while

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- 优点：算法简单，只要求f连续.
- 缺点：使用条件限制较大，收敛速度较慢，且只能求一个根，精度有限.

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若极限$x^*=\lim\limits_{k\to+\infty} x_k$ 不存在，则迭代失败；此时， 可考虑换其它的初值 x_0或采用其它的迭代格式（即构造新的等价形式或新的 ϕ 函数）.

例

代数方程 $x^3 - 2x - 5 = 0$的三种等价方程和迭代格式：

1) $x = \sqrt[3]{2x+5} = \phi(x)$， 迭代格式 $x_{k+1} = \sqrt[3]{2x_k+5}$ $=\phi(x_k)$

2) $x = \frac{x^3-5}{2} = \phi(x)$， 迭代格式 $x_{k+1} = \frac{x_k^3-5}{2}$ $=\phi(x_k)$

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2. 迭代序列的收敛是否与初值x_0有关？

压缩映射定理

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定理：$\phi(x) \in C^1[a,b]$ 满足：

1. $a \leqslant \phi(x) \leqslant b, x \in [a,b]$;
2. $\exists\, 0 < L < 1, s.t$ 对 $\forall x \in [a,b]$ 有 $|\phi'(x)| \leqslant L$ （ϕ称为压缩映射）.

则有：

1. 存在唯一的 x^* 使得 $x^* = \phi(x^*)$ (x^* 称为ϕ的不动点).
2. $\forall x_0 \in [a,b]$，迭代序列$\{x_k\}$收敛，且有误差估计：

$$|x^* - x_k| \leqslant \frac{L^k}{1-L}|x_1 - x_0|$$

Proof

1). 令$\psi(x) = x - \phi(x)$，则$\psi(a) \leq 0$，$\psi(b) \geq 0$. 由介值定理知

$$\exists x^*, s.t\ \psi(x^*) = 0, \Longrightarrow x^* = \phi(x^*)$$

又若$x^{**} = \phi(x^{**})$，则：

$$|x^* - x^{**}| = |\phi(x^*) - \phi(x^{**})| = \left|\phi'(\xi)(x^* - x^{**})\right| \leqslant L\,|x^* - x^{**}|$$

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2). $\forall x_0 \in [a, b]$，有$x_{k+1} - x^* = \phi(x_k) - \phi(x^*) = \phi'(\xi)(x_k - x^*)$. 故

$$|x_{k+1} - x^*| \leqslant L\,|x_k - x^*| \leqslant \cdots \leqslant L^{k+1}\,|x_0 - x^*|$$

所以，对任意$[a, b]$中的初值x_0，迭代序列$\{x_k\}$都收敛到x^*.

误差估计

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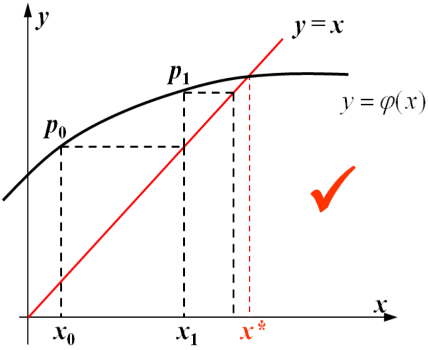
$|x_{k+1}-x_k| = |\phi(x_k)-\phi(x_{k-1})| \leqslant L\,|x_k-x_{k-1}| \leqslant \cdots \leqslant L^k\,|x_1-x_0|$. 故

$$\begin{aligned}
|x_{k+p}-x_k| &\leqslant |x_{k+p}-x_{k+p-1}| + \cdots + |x_{k+1}-x_k| \\
&\leqslant \left(L^{k+p-1}+\cdots+L^k\right)|x_1-x_0| \\
&= \frac{L^k\left(1-L^p\right)}{1-L}\,|x_1-x_0|
\end{aligned}$$

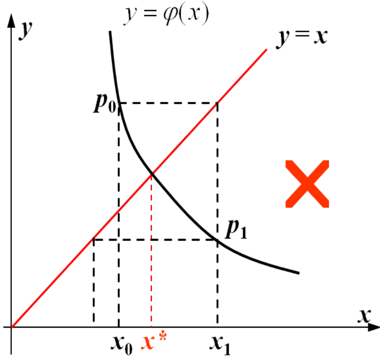
由 p 的任意性，令$p \to +\infty$即得

$$|x^*-x_k| \leqslant \frac{L^k}{1-L}\,|x_1-x_0|$$

迭代法图示—迭代格式 $x_{k+1} = \varphi(x_k)$



1. 迭代成功



2. 迭代失败

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1. How to get such Newton's iteration formula?
2. Does it work or not? Under what condition?
3. Is it fast (i.e., convergent rate)?

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取其线性部分 $L_0(x) = f(x_0) + f'(x_0)(x - x_0)$ **来近似原函数** $f(x)$, 同时用$L_0(x)$的根 $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ 来近似 $f(x)$的根 r .

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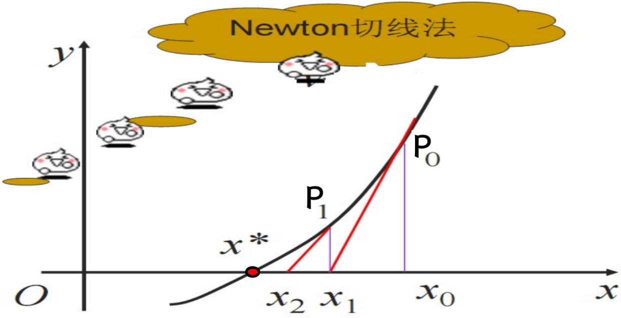
注：当x_k越来越靠近真解（根）时，线性函数$L_k(x)$近似f的效果会越好，得出的近似解也越精确.

Newton迭代的几何解释



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对Newton迭代格式, 有

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- 若x^*为f的p重根，则可得 $\phi'(x^*)=1-\frac{1}{p}$. 此时只需取 x_0 离 x^* 足够近，仍可保证收敛性（为什么?）.

Example

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Found by a Greek engineer and architect Heron about 2000 years ago!

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用牛顿法求方程 $xe^x - 1 = 0$ 在0.5附近的根, 误差精度 $\epsilon = 10^{-5}$.

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k	x_k	$f(x_k)$	$\lvert x_k - x_{k-1}\rvert$
0	0.5	-0.17563936	
1	0.57102044	0.01074751	0.07102044
2	0.56715557	0.00003393	0.00386487
3	0.56714329	0.000000003	0.00001228
4	0.56714329	0.000000003	0.00000000

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定义：$\{x_k\} \to x^*$，$\varepsilon_k = \left|x^* - x_k\right|$. 若$\exists p \geq 1$和正常数$c$，s.t $\lim\limits_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^p} = c$，则称$\{x_k\}$为$p$阶收敛的，也称相应的迭代格式为$p$阶收敛.

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$$x_{k+1} - \alpha = \phi(x_k) - \phi(\alpha) = \phi(\alpha) + (x_k - \alpha)\phi'(\alpha) + \frac{1}{2}(x_k - \alpha)^2\phi''(\xi) - \phi(\alpha)$$

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Recall: 在Newton迭代格式中, 有

$$\phi'(x) = \left(x - \frac{f(x)}{f'(x)}\right)' = \frac{f(x)f''(x)}{\left(f'(x)\right)^2}$$

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Proof.

By assumption, $f(\alpha) = 0 \neq f'(\alpha)$. Denote the error $e_n = x_n - \alpha$, then by the Newton's iteration, we have

$$\begin{aligned} e_{n+1} &= x_{n+1} - \alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \alpha \\ &= e_n - \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}. \end{aligned} \tag{1}$$

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By Taylor's Theorem, we have

$$0 = f(\alpha) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n)$$

$$\Longrightarrow e_n f'(x_n) - f(x_n) = \frac{1}{2} e_n^2 f''(\xi_n) \tag{2}$$

where ξ_n is a number between x_n & α. $\square$

Newton迭代格式单根时的收敛阶

Proof(continued...)

Putting (2) into (1) leads to

$$e_{n+1} = \frac{f''(\xi_n)}{2f'(x_n)} e_n^2 \approx \frac{f''(\alpha)}{2f'(\alpha)} e_n^2 = Ce_n^2 \quad \text{(quad. conv.)} \tag{3}$$

when e_n is small.

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$$x_{k+1} = \phi(x_k) = x_k - p\frac{f(x_k)}{f'(x_k)}$$

是二阶收敛的 (可令 $f(x) = (x - \alpha)^p h(x)$,参见课本P68)。

Example (3)

验证迭代格式 $x_{k+1} = x_k(x_k^2 + 3a)/(3x_k^2 + a) \;\; (k = 0, 1, 2, \cdots)$ 是求$\sqrt{a}$ $(a > 0)$的三阶方法.

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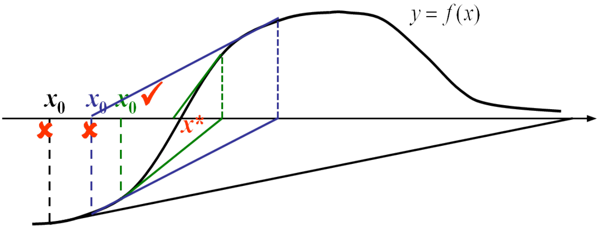

Remark：Newton's method usually assume that its initial guess is sufficiently close to a zero or that the graph of f has a prescribed shape.

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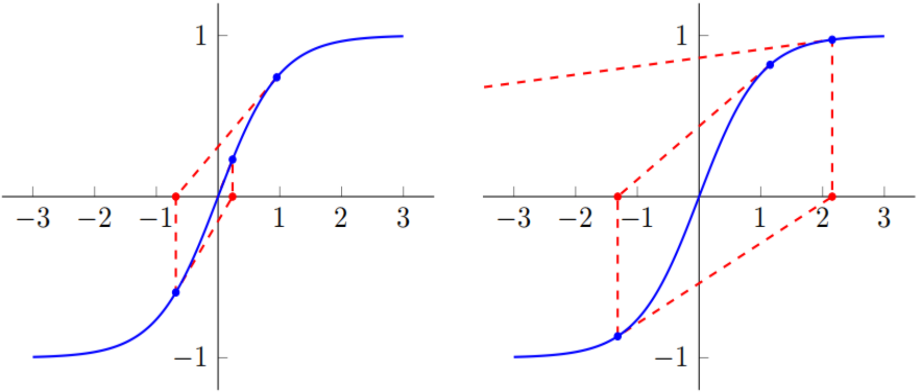
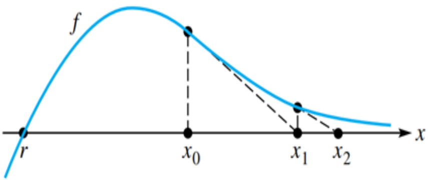
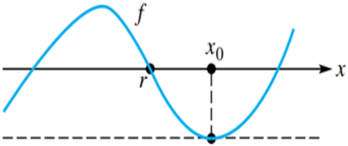


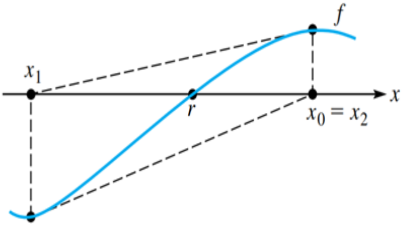
Figure The first iterations in the Newton algorithm for solving $f(x) = 0$, for two starting points: $x^{(1)} = 0.95$ and $x^{(1)} = 1.15$.



(a) Runaway



(b) Flat spot



(c) Cycle

弦截法

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将Newton迭代中的导数用差商$f[x_{k-1},x_k]=\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$代替，得迭代格式

$$x_{k+1}=x_k-f(x_k)\frac{x_k-x_{k-1}}{f(x_k)-f(x_{k-1})}$$

称为弦截法，为二步格式(需两个初始点启动). 单根时，收敛阶约为1.618，比Newton迭代稍慢.

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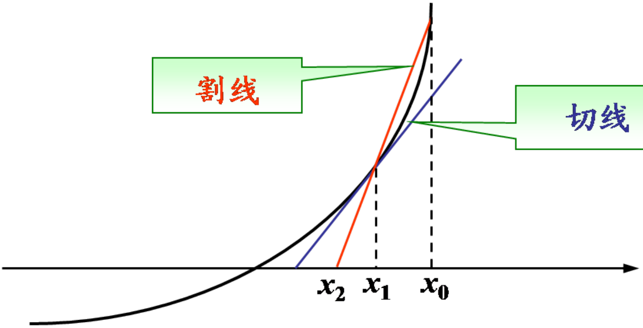

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k	x_k	$f(x)$
0	**1.5**	-0.45
1	**4**	2.3
2	**1.90909**	0.248835
3	**1.65543**	-0.0805692
4	**1.71748**	0.0287456
5	**1.70116**	0.00195902
6	**1.69997**	-0.0000539246
7	**1.7**	9.459×10^{-8}

3.4 非线性方程组的Newton迭代法

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设有二元方程组(x, y为自自变量)

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写成向量形式： $F(w)=0$, 其中

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$$F(w)=\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix},\quad w=(x,y)^T$$

对 f, g 在 (x_0, y_0)作二元Taylor 展开，取其线性部分，得

$$\begin{cases} f(x,y)\approx f(x_0,y_0)+(x-x_0)\frac{\partial f(x_0,y_0)}{\partial x}+(y-y_0)\frac{\partial f(x_0,y_0)}{\partial y}=0 \\ g(x,y)\approx g(x_0,y_0)+(x-x_0)\frac{\partial g(x_0,y_0)}{\partial x}+(y-y_0)\frac{\partial g(x_0,y_0)}{\partial y}=0 \end{cases}$$

令 $\Delta x=x-x_0, \Delta y=y-y_0$， 则

$$J(x_0,y_0)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}=\begin{pmatrix} -f(x_0,y_0) \\ -g(x_0,y_0) \end{pmatrix}.$$

这里，Jacobi(雅克比)矩阵

$$J(x_0, y_0) \doteq \begin{pmatrix} \frac{\partial f}{\partial x}\big|_{(x_0,y_0)} & \frac{\partial f}{\partial y}\big|_{(x_0,y_0)} \\ \frac{\partial g}{\partial x}\big|_{(x_0,y_0)} & \frac{\partial g}{\partial y}\big|_{(x_0,y_0)} \end{pmatrix}.$$

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若$det(J(x_0, y_0)) \neq 0$, 可解出 $\Delta x, \Delta y$. 令

$$w_1 \doteq \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \doteq \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} x_0 + \Delta x \\ y_0 + \Delta y \end{pmatrix}.$$

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$$w_1 \doteq \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \doteq \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} x_0 + \Delta x \\ y_0 + \Delta y \end{pmatrix}.$$

同理，再对 f, g 在 (x_1, y_1)作二元Taylor 展开，并取其线性部分。 若$det(J(x_1, y_1)) \neq 0$, 可解出 $\Delta x = x - x_1, \Delta y = y - y_1$, 进而得到

$$w_2 \doteq \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \doteq \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} x_1 + \Delta x \\ y_1 + \Delta y \end{pmatrix}.$$

继续做下去...
每次迭代先解一个关于 $\Delta x \doteq x - x_k, \Delta y \doteq y - x_k$ 的二元方程组

$$J(x_k, y_k) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -f(x_k, y_k) \\ -g(x_k, y_k) \end{pmatrix}.$$

进而得到新的迭代点

$$w_{k+1} \doteq \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} \doteq \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} x_k + \Delta x \\ y_k + \Delta y \end{pmatrix}.$$

Example (3.6)

求非线性方程组

$$\begin{cases} f_1(x,y) \doteq 4 - x^2 - y^2 = 0 \\ f_2(x,y) \doteq 1 - e^x - y = 0 \end{cases}, \quad \text{取初始值} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.7 \end{pmatrix}.$$

$$J(x,y)=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix}=\begin{pmatrix}-2x & -2y\\ -e^x & -1\end{pmatrix}$$

$$J(x_0,y_0)=\begin{pmatrix}-2 & 3.4\\ -2.71828 & -1\end{pmatrix},\quad \begin{pmatrix}f_1(x_0,y_0)\\ f_2(x_0,y_0)\end{pmatrix}=\begin{pmatrix}0.11\\ -0.01828\end{pmatrix}$$

解方程 $$\begin{cases}-2\Delta x+3.4\Delta y=-0.11\\ -2.71828\Delta x-\Delta y=0.01828\end{cases}$$

$$J(x_k,y_k)\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}=\begin{pmatrix}-f(x_k,y_k)\\ -g(x_k,y_k)\end{pmatrix}$$

得 $$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}=\begin{pmatrix}0.004256\\ -0.029849\end{pmatrix}$$

$$w_1=\begin{pmatrix}x_0\\ y_0\end{pmatrix}+\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}=\begin{pmatrix}1.004256\\ -1.729849\end{pmatrix}$$

继续做下去，直到 $\max(|\Delta x|,|\Delta y|)<10^{-5}$ 时停止.

一般非线性方程组的Newton迭代

设方程组为 $\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases}$，写成向量形式为：

$$F(X) = 0,\ \text{其中} F = (f_1, f_2, \cdots, f_n)^T,\ X = (x_1, x_2, \cdots, x_n)^T$$

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仿照单变量Newton法，取初值$X_0 = (x_{01}, x_{02}, \cdots, x_{0n})^T$，将$\{f_i\}_{i=1}^n$在$X_0$处进行Taylor展开后， 分别取相应的线性部分来近似每一个f_i，得：

$$\begin{cases} f_1(X_0) + \frac{\partial f_1}{\partial x_1}(x_1 - x_{01}) + \cdots + \frac{\partial f_1}{\partial x_n}(x_n - x_{0n}) = 0 \\ f_2(X_0) + \frac{\partial f_2}{\partial x_1}(x_1 - x_{01}) + \cdots + \frac{\partial f_2}{\partial x_n}(x_n - x_{0n}) = 0 \\ \vdots \\ f_n(X_0) + \frac{\partial f_n}{\partial x_1}(x_1 - x_{01}) + \cdots + \frac{\partial f_n}{\partial x_n}(x_n - x_{0n}) = 0 \end{cases}$$

写成向量形式即为

$$F(X_0) + J_F(X_0)(X - X_0) = 0 \qquad (\,*\,)$$

其中

$$J_F(X) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

表示F 的Jacobi矩阵.

方程组$(\,*\,)$的解为$X_1 = X_0 - J_F^{-1}(X_0)F(X_0)$，再将$\{f_i\}_{i=1}^n$在$X_1$处Taylor展开.

同理可得推广的Newton迭代格式为：

$$X_{k+1} = X_k - \frac{F(X_k)}{F'(X_k)} = X_k - \Big(J_F(X_k)\Big)^{-1} F(X_k)$$

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在实际中，一般通过解以下线性代数方程组

$$J_F(X_k)(X_{k+1} - X_k) = -F(X_k)$$

来求X_{k+1}.